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ARTICLE *in* LINEAR ALGEBRA AND ITS APPLICATIONS · AUGUST 2013

Impact Factor: 0.94 · DOI: 10.1016/j.laa.2012.08.005

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AN OPERATOR INEQUALITY AND ITS CONSEQUENCES

M. S. MOSLEHIAN, J. MIĆIĆ AND M. KIAN

Dedicated to Professor Harm Bart on the occasion of his 70th birthday

Abstract. Let f be a continuous convex function on an interval J, let A, B, C, D be self-adjoint operators acting on a Hilbert space with spectra contained in J such that $A + D = B + C$ and $A \leq m \leq B, C \leq M \leq D$ for two real numbers $m < M$, and let Φ be a unital positive linear map on $\mathbb{B}(\mathscr{H})$. We prove the inequality

$$f(\Phi(B)) + f(\Phi(C)) \leq \Phi(f(A)) + \Phi(f(D)).$$

and apply it to obtain several inequalities such as the Jensen–Mercer operator inequality and the Petrović operator inequality.

1. Introduction and Preliminaries

Throughout the paper $\mathbb{B}(\mathscr{H})$ stands for the C^*-algebra of all bounded linear operators on a complex Hilbert space $\mathscr{H}$ and I denotes the identity operator. The real subspace of $\mathbb{B}(\mathscr{H})$ consisting of all self-adjoint operators on $\mathscr{H}$ is denoted by $\mathbb{B}(\mathscr{H})_h$. An operator A is said to be positive (denoted by $A \geq 0$) if $\langle Ax, x\rangle \geq 0$ for all $x \in \mathscr{H}$. If, in addition, A is invertible, then it is called strictly positive (denoted by $A > 0$). By $A \geq B$ we mean that $A - B$ is positive, while $A > B$ means that $A - B$ is strictly positive. A map Φ on $\mathbb{B}(\mathscr{H})$ is said to be positive if $\Phi(A) \geq 0$ for each $A \geq 0$ and is called unital if $\Phi(I) = I$.

Let f be a continuous real valued function defined on an interval J. The function f is called operator monotone if $A \geq B$ implies $f(A) \geq f(B)$ for all $A, B \in \mathbb{B}(\mathscr{H})_h$ with spectra in J. A function f is called operator convex on J if $f(\lambda A + (1-\lambda)B) \leq \lambda f(A) + (1-\lambda) f(B)$ or all $A, B \in \mathbb{B}(\mathscr{H})_h$ with spectra in J and all $\lambda \in [0, 1]$. If the function f is operator convex, then the so-called Jensen operator inequality $f(\Phi(A)) \leq \Phi(f(A))$ holds for any unital positive linear map Φ on $\mathbb{B}(\mathscr{H})$ and any $A \in \mathbb{B}(\mathscr{H})_h$ with spectrum contained in J. Many other

2010 *Mathematics Subject Classification.* 15A60, 47A63,47A64, 26D15.

Key words and phrases. convex function, positive linear map, Jensen–Mercer operator inequality, Petrović operator inequality.

versions of the Jensen operator inequality can be found in [6]. Among them, the Jensen–Mercer operator inequality [8] reads as follows:

$$f\left(M+m-\sum_{i=1}^{n}\Phi_i(A_i)\right)\le f(M)+f(m)-\sum_{i=1}^{n}\Phi_i(f(A_i)),$$

where f is a continuous convex function on an interval $[m,M]$, $\Phi_1,\cdots,\Phi_n$ are positive linear maps on $\mathbb{B}(\mathscr{H})$ with $\sum_{i=1}^{n}\Phi_i(I)=I$ and $A_1,\cdots,A_n\in\mathbb{B}(\mathscr{H})_h$ with spectra contained in $[m,M]$.

Another interesting inequality is the Petrović inequality (see e.g. [10, page 152]). It states that if $f:[0,\infty)\to\mathbb{R}$ is a continuous convex function and $A_1,\cdots,A_n$ are positive operators such that $\sum_{i=1}^{n}A_i=MI$ for some scalar $M>0$, then

$$\sum_{i=1}^{n}f(A_i)\le f\left(\sum_{i=1}^{n}A_i\right)+(n-1)f(0).$$

If $f:[0,\infty)\to[0,\infty)$ is a convex function and $f(0)\le 0$, then

$$f(a)+f(b)\le f(a+b) \tag{1.1}$$

for all scalars $a,b\ge 0$. However, if the scalars a,b are replaced by two positive operators, this inequality may not hold. There have been many interesting works devoted to obtain norm or operator extensions of inequality (1.1) and other subadditivity inequalities. Ando and Zhan [1] showed that if f is a nonnegative operator monotone function on the interval $[0,\infty)$, then

$$|||f(A+B)|||\le|||f(A)+f(B)||| \tag{1.2}$$

for all unitarily invariant norms $|||\cdot|||$ and all matrices $A,B\ge 0$. Bourin and Uchiyama [5] extended Ando-Zhan's result by showing that if $A,B\ge 0$ and f is a non-negative concave function on $[0,\infty)$, then (1.2) holds for all unitarily norms. Also Aujla and Bourin [3] showed that if $A,B\ge 0$ and $f:[0,\infty)\to[0,\infty)$ is a monotone concave function, then there exist unitaries U, V such that

$$f(A+B)\le Uf(A)U^*+Vf(B)V^*.$$

The reader is referred to [4, 11, 7, 12, 9, 2] and references therein for recent information on the subject.

In their study of Hadamard's inequalities for co-ordinated convex functions, Hwang, Tseng and Yang proved that if $f:[a,b]\to\mathbb{R}$ is a convex function and

$a \le y_1 \le x_1 \le x_2 \le y_2 \le b$ and $x_1 + x_2 = y_1 + y_2$, then

$$f(x_1) + f(x_2) \le f(y_1) + f(y_2).$$

In this paper, we extend this inequality to operators acting on a Hilbert space and apply it to obtain a series of operator inequalities including the Jensen–Mercer operator inequality, the Petrović operator inequality.

2. Results

We start this section with our main result.

Theorem 2.1. *Let f be a continuous convex function on an interval J. Let $A, B, C, D \in \mathbb{B}(\mathscr{H})_h$ with spectra contained in J such that $A + D = B + C$ and $A \le m \le B, C \le M \le D$ for two real numbers $m < M$. If Φ is a unital positive linear map on $\mathbb{B}(\mathscr{H})$, then*

$$f(\Phi(B)) + f(\Phi(C)) \le \Phi(f(A)) + \Phi(f(D)). \tag{2.1}$$

If f is concave on J, then inequality (2.1) is reversed.

Proof. If $t \in [m, M]$, then $0 \le \frac{M-t}{M-m} \le 1$. Since f is convex on $[m, M]$, we have

$$f(t) = f\left(\frac{M-t}{M-m}m + \frac{t-m}{M-m}M\right) \le \frac{M-t}{M-m}f(m) + \frac{t-m}{M-m}f(M). \tag{2.2}$$

If $t \in J \setminus (m, M)$, then $t \le m$ or $M \le t$. Suppose that $t \le m$ (the case when $M \le t$ can be verified similarly). Therefore $0 \le \frac{M-m}{M-t} \le 1$ and

$$f(m) = f\left(\frac{M-m}{M-t}t + \frac{m-t}{M-t}M\right) \le \frac{M-m}{M-t}f(t) + \frac{m-t}{M-t}f(M),$$

whence

$$f(t) \ge \frac{M-t}{M-m}f(m) + \frac{t-m}{M-m}f(M) \tag{2.3}$$

for all $t \in J \setminus (m, M)$. Since $A \le m$ and $D \ge M$, using functional calculus to inequality (2.3) we obtain

$$f(A) \ge \frac{M-A}{M-m}f(m) + \frac{A-m}{M-m}f(M) \tag{2.4}$$

and

$$f(D) \ge \frac{M-D}{M-m}f(m) + \frac{D-m}{M-m}f(M). \tag{2.5}$$

Applying the positive linear map Φ to both sides of (2.4) and (2.5) we get

$$\Phi(f(A)) \ge \frac{M-\Phi(A)}{M-m}f(m) + \frac{\Phi(A)-m}{M-m}f(M) \tag{2.6}$$

and

$$\Phi(f(D)) \geq \frac{M-\Phi(D)}{M-m}f(m) + \frac{\Phi(D)-m}{M-m}f(M). \tag{2.7}$$

Similarly, taking into account that $m \leq \Phi(B) \leq M$ and $m \leq \Phi(C) \leq M$ and using functional calculus to inequality (2.2) we obtain

$$f(\Phi(B)) \leq \frac{M-\Phi(B)}{M-m}f(m) + \frac{\Phi(B)-m}{M-m}f(M) \tag{2.8}$$

and

$$f(\Phi(C)) \leq \frac{M-\Phi(C)}{M-m}f(m) + \frac{\Phi(C)-m}{M-m}f(M). \tag{2.9}$$

Adding two inequalities (2.8) and (2.9) we obtain

$$\begin{aligned}
f(\Phi(B)) + f(\Phi(C)) &\leq \frac{M-\Phi(B)}{M-m}f(m) + \frac{\Phi(B)-m}{M-m}f(M) \\
&\quad + \frac{M-\Phi(C)}{M-m}f(m) + \frac{\Phi(C)-m}{M-m}f(M) \\
&= \frac{2M-\Phi(B+C)}{M-m}f(m) + \frac{\Phi(B+C)-2m}{M-m}f(M) \\
&= \frac{2M-\Phi(A+D)}{M-m}f(m) + \frac{\Phi(A+D)-2m}{M-m}f(M) \\
&\qquad\qquad\qquad\qquad \text{(by } A+D=B+C\text{)} \\
&= \frac{M-\Phi(A)}{M-m}f(m) + \frac{\Phi(A)-m}{M-m}f(M) \\
&\quad + \frac{M-\Phi(D)}{M-m}f(m) + \frac{\Phi(D)-m}{M-m}f(M) \\
&\leq \Phi(f(A)) + \Phi(f(D)) \qquad \text{(by (2.6) and (2.7))}
\end{aligned}$$

□

We give an example to clarify the situation in Theorem 2.1.

Example 2.2. Let the function f be defined on $[0,\infty)$ by $f(t) = t^3$ and the unital positive linear map $\Phi : \mathscr{M}_2(\mathbb{C}) \to \mathscr{M}_2(\mathbb{C})$ be defined by $\Phi(A) = (\frac{1}{2}tr(A))I$ for all Hermitian matrices $A \in \mathscr{M}_2(\mathbb{C})$. If

$$A = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 4 & 1 \\ 1 & 4 \end{pmatrix}, \quad C = \begin{pmatrix} 6 & -1 \\ -1 & 7 \end{pmatrix}, \quad D = \begin{pmatrix} 9 & 1 \\ 1 & 10 \end{pmatrix},$$

then

$$0 \leq A < 3I \leq B \leq C \leq 8I < D \qquad \text{and} \quad A+D = B+C,$$

whence

$$f(\Phi(B)) + f(\Phi(C)) = 338.625I \lneq 897I = \Phi(f(A)) + \Phi(f(D)).$$

This shows that inequality (2.1) can be strict.

More generally, the next corollary gives other versions of inequality (2.1). The proof is similar to that of Theorem 2.1 and we omit it.

Corollary 2.3. *Let f be a continuous convex function on an interval J. Let $A_i, B_i, C_i, D_i \in \mathbb{B}(\mathscr{H})_h \quad (i = 1, \cdots, n)$ with spectra contained in J such that $A_i + D_i = B_i + C_i$ and $A_i \leq m \leq B_i \leq M \leq D_i$ and $A_i \leq m \leq C_i \leq M \leq D_i \quad (i = 1, \cdots, n)$. Let $\Phi_1, \cdots, \Phi_n$ be positive linear maps on $\mathbb{B}(\mathscr{H})$ with $\sum_{i=1}^n \Phi_i(I) = I$. Then*

(1) $f\left(\sum_{i=1}^n \Phi_i(B_i)\right) + f\left(\sum_{i=1}^n \Phi_i(C_i)\right) \leq \sum_{i=1}^n \Phi_i(f(A_i)) + \sum_{i=1}^n \Phi_i(f(D_i))$;

(2) $\sum_{i=1}^n \Phi_i(f(B_i)) + \sum_{i=1}^n \Phi_i(f(C_i)) \leq f\left(\sum_{i=1}^n \Phi_i(A_i)\right) + f\left(\sum_{i=1}^n \Phi_i(D_i)\right)$;

(3) $\sum_{i=1}^n \Phi_i(f(B_i)) + f\left(\sum_{i=1}^n \Phi_i(C_i)\right) \leq f\left(\sum_{i=1}^n \Phi_i(D_i)\right) + \sum_{i=1}^n \Phi_i(f(A_i))$.

The Jensen–Mercer operator inequality follows directly from Theorem 2.1.

Corollary 2.4. [8, Theorem 1] *Let $\Phi_1, \cdots, \Phi_n$ be positive linear maps on $\mathbb{B}(\mathscr{H})$ with $\sum_{i=1}^n \Phi_i(I) = I$ and $B_1, \cdots, B_n \in \mathbb{B}(\mathscr{H})_h$ with spectra contained in $[m, M]$. If f is a continuous convex function on $[m, M]$, then*

$$f\left(m + M - \sum_{i=1}^n \Phi_i(B_i)\right) \leq f(m) + f(M) - \sum_{i=1}^n \Phi_i(f(B_i)).$$

Proof. Clearly $m \leq B_i \leq M \quad (i = 1, \cdots, n)$. Set $C_i = M + m - B_i \quad (i = 1, \cdots, n)$. Then $m \leq C_i \leq M$ and $B_i + C_i = m + M \quad (i = 1, \cdots, n)$. Applying inequality (3) of Corollary 2.3 when $A_i = mI$ and $D_i = MI$ we obtain

$$\sum_{i=1}^n \Phi_i(f(B_i)) + f\left(\sum_{i=1}^n \Phi_i(C_i)\right) \leq f(m) + f(M),$$

which is the desired inequality. □

The next result provides an extension of Petrović inequality.

Corollary 2.5. *Let $A, D, B_i \in \mathbb{B}(\mathscr{H})_h$ $(i = 1, \cdots, n)$ with spectra contained in an interval J such that $A + D = \sum_{i=1}^n B_i$ and $A \le m \le B_i \le M \le D$ $(i = 1, \cdots, n)$ for two real numbers $m < M$. If f is convex on J, then*

$$\sum_{i=1}^n f(B_i) \le (n-1) f\left(\frac{1}{n-1} A\right) + f(D).$$

The next result is a Jensen operator inequality for continuous convex functions. First define the subset Ω of $\mathbb{B}_h(\mathscr{H}) \times \mathbb{B}_h(\mathscr{H})$ by

$$\Omega = \left\{ (A, B) \mid A \le m \le \frac{A+B}{2} \le M \le B, \quad \text{for some} \quad m, M \in \mathbb{R} \right\}.$$

The authors would like to pose the following problem that is interesting on its own right.

Problem 2.6. Is there any characterization of Ω for either matrices or operators acting on infinite dimensional Hilbert spaces?

Corollary 2.7. *If f is a continuous convex function on an interval J, then*

$$f(\lambda A + (1-\lambda) D) \le \lambda f(A) + (1-\lambda) f(D) \tag{2.10}$$

for all $(A, D) \in \Omega$ with spectra contained in J and all $\lambda \in [0, 1]$. If f is concave, then inequality (2.10) is reversed.

Proof. Letting Φ be the identity map and putting $C = B = \frac{A+D}{2}$ in inequality (2.1), we get

$$f\left(\frac{A+D}{2}\right) \le \frac{f(A) + f(D)}{2}$$

for any $(A, D) \in \Omega$ with spectra contained in J which implies inequality (2.10). □

Note that the existence of scalars $m < M$ is essential in Corollary 2.7, i.e., inequality (2.10) may not hold if $A, B \notin \Omega$.

Example 2.8. Consider the convex function $f(t) = t^3$ on $[0, \infty)$. Putting

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad B = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 4 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

we have $0 \le A \le B$. There is no scalar m such that $A \le m \le \frac{A+B}{2}$. Now

$$\left(\frac{A+B}{2}\right)^3 = \begin{pmatrix} 6 & 14 & 0 \\ 14 & 34 & 0 \\ 0 & 0 & 3.375 \end{pmatrix} \not\le \begin{pmatrix} 6 & 15 & 0 \\ 15 & 43 & 0 \\ 0 & 0 & 4.5 \end{pmatrix} = \frac{A^3+B^3}{2},$$

shows that not only f is not operator convex but also the existence of scalars m, M are essential in definition of Ω.

Generally, inequality (1.1) would be false if we replace scalars a, b with two positive operators. To see this consider the convex function $f(t) = t^3$ on $[0,\infty)$ and two positive matrices A, B in Example 2.8. The following corollary gives an operator version of (1.1). The reader may compare it with [7, Theorem 2.2].

Corollary 2.9. *If $f : [0,\infty) \to [0,\infty)$ is a continuous convex function with $f(0) \le 0$, then*

$$f(A) + f(B) \le f(A+B)$$

for all strictly positive operators A, B for which $A \le M \le A+B$ and $B \le M \le A+B$ for some scalar M.

Proof. It follows directly from Corollary 2.5. □

Theorem 2.10. *Let f be a continuous function on an interval J. Let $A, B, C, D \in \mathbb{B}(\mathscr{H})_h$ with spectra contained in J such that $A \le m \le B, C \le M \le D$ for two real numbers $m \le M$. If f is convex and one of the following conditions*

$$\begin{aligned} &\text{(i)} \quad B + C \le A + D \quad \textit{and} \quad f(m) \le f(M) \\ &\text{(ii)} \quad A + D \le B + C \quad \textit{and} \quad f(M) \le f(m) \end{aligned}$$

is satisfied, then

$$f(B) + f(C) \le f(A) + f(D). \tag{2.11}$$

If f is concave and one of the following conditions

$$\begin{aligned} &\text{(iii)} \quad B + C \le A + D \quad \textit{and} \quad f(M) \le f(m) \\ &\text{(iv)} \quad A + D \le B + C \quad \textit{and} \quad f(m) \le f(M) \end{aligned}$$

is satisfied, then inequality (2.11) is reversed.

Proof. Let f be convex and (i) is valid. It follows from (2.2) that

$$f(B) \leq \frac{f(M)-f(m)}{M-m}B + \frac{f(m)M-f(M)m}{M-m}$$

and

$$f(C) \leq \frac{f(M)-f(m)}{M-m}C + \frac{f(m)M-f(M)m}{M-m}.$$

Summing above inequalities we get

$$\begin{aligned} f(B)+f(C) &\leq \frac{f(M)-f(m)}{M-m}(B+C) + 2\,\frac{f(m)M-f(M)m}{M-m} \\ &\leq \frac{f(M)-f(m)}{M-m}(A+D) + 2\,\frac{f(m)M-f(M)m}{M-m} \qquad \text{(by (i))} \\ &= \frac{f(M)-f(m)}{M-m}A + \frac{f(m)M-f(M)m}{M-m} \\ &\quad + \frac{f(M)-f(m)}{M-m}D + \frac{f(m)M-f(M)m}{M-m} \\ &\leq f(A)+f(D) \qquad \text{(by (2.4) and (2.5))} \end{aligned}$$

The other cases are verified similarly. □

As an immediate consequence of Theorem 2.10, we have

Corollary 2.11. *Let f and A, B, C, D be as in Theorem 2.10. If f is convex and one of the following conditions*

$$\text{(i)} \quad B+C \leq A+D \quad \textit{and} \quad f(m) \leq f(M)$$

$$\text{(ii)} \quad A+D \leq B+C \quad \textit{and} \quad f(M) \leq f(m)$$

is satisfied, then

$$f(B)+f(C) \leq g(A)+g(D),$$

for every continuous function $g \geq f$ on J and

$$g(B)+g(C) \leq f(A)+f(D),$$

or every continuous function $g \leq f$ on J.

Applying the above corollary to the power functions we get

Corollary 2.12. *Let $A, B, C, D \in \mathbb{B}(\mathscr{H})_h$ such that $I \leq A \leq m \leq B, C \leq M \leq D$ for two real numbers $m \leq M$. If one of the following conditions*

$$\text{(i)} \quad B+C \leq A+D \quad \textit{and} \quad p \geq 1$$

$$\text{(ii)} \quad A+D \leq B+C \quad \textit{and} \quad p \leq 0$$

is satisfied, then

$$B^p + C^p \leq A^q + D^q$$

for each $q \geq p$.

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